NONLINEAR **AERODYNAMIC** GLOBAL **MODEL IDENTIFICATION** USING **GRAM-SCHMIDT ORTHOGONALIZATION**

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Abstract

This paper discusses a simple technique to identify global models for nonlinear aerodynamic force and moment coefficients of aircraft using multivariate orthogonal functions. Classical Gram-Schmidt procedure and Predicted Squared Error metric are used to generate the orthogonal functions. Global models for the F-16 aircraft are identified from a simplified subsonic (Mach < 0.6) wind tunnel database available in open literature. The identified models are compared with those found in literature for the same wind tunnel database and conclusions are drawn.

Keywords : *Global model, aerodynamic coefficients, orthogonal functions, multivariate polynomials, Predicted Squared Error, Gram-Schmidt method*

Nomenclature

a	= element of matrix A
A	= MxM unit upper triangular matrix
b_j	= j^{th} ordinary polynomial function parameter
B	= ordinary polynomial function parameter vector
C_x, C_y, C_z	= aerodynamic force coefficients
C_{xq}, C_{zq}	= force derivatives due to pitch rate
C_l, C_m, C_n	= aerodynamic moment coefficients
J	= least squares cost function
K	= weighting factor used in OFP term
MSE	= Mean Squared Error
M	= number of retained orthogonal functions
N	= number of sample times
OFP	= Over Fit Penalty
p_j	= j^{th} column vector of regression matrix P
P	= NxM matrix of ordinary polynomial functions
PSE	= predicted squared error, PSE = MSE + OFP
q	= pitch rate (rad or deg/sec)
w_j	= j^{th} column of matrix W
W	= NxM matrix with mutually orthogonal columns
y_i	= i^{th} value of the dependent variable
$\bar{y}$	= average value of y_i
y	= dependent variable vector
α, β	= angle of attack, sideslip angle (rad or deg)
δ_j	= reduction in J contributed by j^{th} orthogonal function
γ_j	= j^{th} orthogonal function parameter
Γ	= orthogonal function parameter vector
σ_0^2	= maximum prediction MSE
σ_{max}^2	= *a priori* upper bound estimate of prediction MSE
ξ_j	= j^{th} element of modeling error vector
Ξ	= modelling error vector

Superscripts

^	= estimate
T	= transpose
-1	= matrix inverse

Introduction

Control system design, simulation and optimizatio[n] require a compact analytical description of the aircra[ft] dynamics. The analytical models for aerodynamic for[ce] and moment coefficients can either be "local" or "global[".] The local models are valid only over a limited portion [of] the flight envelope while the global models cover a wid[e] range of the flight envelope, and as such are more usef[ul] for dynamic analysis. The process of simulation, cont[rol] design and aerodynamic analysis can generally be handl[ed] more effectively by replacing multiple local models w[ith] a single global model. For best results, the global nonlin[ear] aerodynamic model has to be compact, with the minim[um] possible number of terms. At the same time, it should ha[ve]

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the ability to capture the aerodynamic nonlinearities over an extended portion of the flight envelope.

Global models, generally expressed as polynomials in independent variables, are easy to update from flight test data. They provide valuable understanding of the underlying physical phenomena, which otherwise can be obscure in the wind tunnel database. Also the analytical models have smooth gradients, that is useful in generating local linear models.

In the past, several techniques were developed for generating global models from wind tunnel database spanning a wide range of independent variables like the angle of attack, sideslip angle and Mach number. These techniques include the least squares linear regression [1, 2], splines in one or two independent variables [3, 4] and splines in association with stepwise regression [5]. More recently, neural networks using radial basis functions have been used to model the wind tunnel database [6]. However, none of these techniques addresses the model structure determination adequately. In the classical least squares method, the model structure determination and the parameter estimation are coupled. The spline functions and neural network techniques offer no clear insight into the physical relationship between the dependent and independent variables. Further, any increase in the number of independent variables or in the range of independent variables complicates the model structure determination problem leading to unsatisfactory results.

Recently, application of nonlinear multivariate orthogonal least squares modeling technique to estimate global models from wind tunnel data has been demonstrated [7-10]. The technique generates nonlinear orthogonal modeling functions from the independent variable data using the algorithm described in [11]. These orthogonal functions, along with a Predicted Squared Error (PSE) metric, are used to determine appropriate model structure of the aerodynamic coefficients [12]. The identified orthogonal functions are eventually converted into multivariate ordinary polynomials in the independent variables. The use of orthogonal functions decouples the least squares problem and the model structure determination problem becomes easier. This allows for easy upgradation of the model with the available data.

The technique described in [11] is used to identify global models for the vertical force coefficient (C_Z) of the F-18 High Angle of Attack Research Vehicle (HARV) and aerodynamic coefficients of the F-16 aircraft [8, 9]. The algorithm generates the orthogonal functions in a sequential manner using a set of unique positive integers to keep track of the order of the generated orthogonal functions. This procedure, though well defined, is involved and not easy to implement. A simpler two-step approach to generate orthogonal functions using the classical Gram-Schmidt method is presented in [13], and is used in a wind tunnel experiment to characterize the aerodynamic and propulsive forces and moments of a research model airplane FASER (Free-flying Airplane for Sub-scale Experimental Research) [13].

The present work uses the technique of [13] to generate global models from F-16 wind tunnel data [14]. The global models so obtained are compared with the ones given in [9], for the same wind tunnel database. It is shown that the approach used in the present work is adequate for generating nonlinear aerodynamic global models of F-16 aircraft.

The PSE metric, which is a sum of the conventional Mean Squared Error (MSE) metric and the Over-Fit Penalty (OFP), is used along with the Gram-Schmidt method to arrive at the number of terms to be included in the multivariate polynomial. The OFP is related to the estimated output variance (σ_0^2) from the wind tunnel measurements. Although the PSE concept is very rational and elegant, there appears to be some ambiguity in the literature in assigning proper weighting to OFP in the expression for PSE. The effect of weighting factors on OFP in determining the global models of the aerodynamic coefficients of F-16 aircraft is also discussed.

Theoretical Development

Assume that the analytical model of an aerodynamic force or moment coefficient can be expressed in the form of a truncated multivariable power series in independent variables. In case, the aerodynamic coefficients that are functions of a single variable, say a then :

$$C_1(\alpha) = \sum_{i=0}^{k} b_i\alpha^i, \quad k = 0, 1, \ldots$$

$$= b_0 + b_1\alpha + b_2\alpha^2 + b_3\alpha^3 + \ldots + b_k\alpha^k \tag{1}$$

The aerodynamic coefficients that are functions of two independent variables, say α and β can be expressed in two basic forms [15] :

$$C_2(\alpha,\beta) = \sum_{i=0}^{k}\sum_{j=0}^{l} b_{ij}\alpha^i\beta^j, \quad k = 0,1,2,\ldots, \quad l = 0,1,2,\ldots$$

or

$$C_2(\alpha,\beta) = \sum_{i=0}^{i+j\le k}\sum_{j=0} b_{ij}\alpha^i\beta^j, \quad k = 0,1,2,\ldots \tag{2}$$

The latter form is used in the present work :

$$C_2(\alpha,\beta) = b_{00} + b_{10}\alpha + b_{01}\beta + b_{20}\alpha^2 + b_{11}\alpha\beta$$
$$+ b_{02}\beta^2 + b_{30}\alpha^3 + \ldots \tag{3}$$

The aerodynamic coefficients that are functions of more than two variables can also be expressed in a similar fashion.

Orthogonal Least Squares Estimation

Let y represent an N-dimensional vector of measured values of an aerodynamic coefficient. Then,

$$y_j = \sum_{k=1}^{M} p_{kj} b_k + \xi_j, \quad j = 1,2,\ldots,N \tag{4}$$

Equation (4) can be written in the matrix form as

$$y = PB + \Xi \tag{5}$$

where $y = [y_1, y_2, \ldots y_N]^T$ is the output vector, $B = [b_1, b_2, \ldots b_M]^T$ is the parameter vector, $\Xi = [\xi_1, \xi_2, \ldots, \xi_N]^T$ is the residual error, and $P = [p_1, p_2, \ldots, p_N]$ is the (NxM) regression matrix with columns $p_i = [p_i(1), p_i(2), \ldots, p_i(N)]^T$, $i = 1, 2, \ldots, M$.

The regression matrix P is orthogonalized using the classical Gram-Schmidt procedure. The orthogonal decomposition of P is given by

$$P = WA \tag{6}$$

where $A = \{a_{ij}\}$ is an MxM upper triangular matrix and $W = [w_1, w_2, \ldots, w_M]$ is an NxM matrixd with orthogonal columns that satisfy the relationship

$$w_i^T w_j = 0 \quad \text{for } i \ne j, \quad i,j = 1,2,\ldots,M \tag{7}$$

Equation (5) can now be expressed as

$$y = (PA^{-1})(AB) + \Xi + W\Gamma + E \tag{8}$$

where AB = Γ, and Γ is an auxiliary vector given by

$$\Gamma = [\gamma_1, \gamma_2, \ldots, \gamma_M]^T \tag{9}$$

Minimizing the cost function

$$J = (y - W\Gamma)^T (y - W\Gamma) \tag{10}$$

gives the least-squares estimate for Γ

$$\hat{\Gamma} = (W^T W)^{-1} W^T y \tag{11}$$

The k^{th} element of the estimated vector $\hat{\Gamma}$ is given by,

$$\hat{\gamma}_k = \frac{w_k^T y}{w_k^T w_k}, \quad k = 1, 2, \ldots, M \tag{12}$$

Equation (12) shows that when w_k arc orthogonal, each γ_k depends only on the measured values of the dependent variable y, and the corresponding orthogonal function w_i. The model parameter vector $B = [b_1, b_2, \ldots, b_m]^T$ can then be calculated from the equation $AB = \Gamma$ through back substitution.

Model Structure Selection

Using Equations (8) and (10), the cost function can be expressed as

$$J = y^T y - 2\sum_{j=1}^{M} \gamma_j w_j^T y + \sum_{j=1}^{M}\sum_{i=1}^{M} \gamma_j \gamma_i w_j^T w_i$$

Using the orthogonality of the functions W_i given in Eq. (7),

$$\hat{J} = y^T y - 2\sum_{j=1}^{M} \hat{\gamma}_j^2 w_j^T w_j + \sum_{j=1}^{M} \hat{\gamma}_j^2 w_j^T w_j = y^T y - \sum_{j=1}^{M} \hat{\gamma}_j^2 w_j^T w_j$$

where $\hat{J}$ is used in place of J because the estimates of y_j are used.

Using Eq. (12),

$$\hat{J} = y^T y - \sum_{j=1}^{M} \frac{(w_j^T y)^2}{(w_j^T w_j)} = y^T y - \sum_{j=1}^{M} \delta_j \tag{16}$$

$$\text{where, } \delta_j = \frac{(w_j^T y)^2}{(w_j^T w_j)} \tag{17}$$

Eq. (16) shows how each orthogonal term w_j reduces the cost function by an amount δ_j. This decouples the least-squares estimation problem, and makes it possible to rank each orthogonal modeling function in terms of its ability to reduce the least-squares model fit to the data, regardless of other orthogonal modeling functions already included in the model.

The Predicted Squared Error (PSE), is used to select the minimum number (M) of orthogonal functions to be included in the global model [12]:

$$PSE = \frac{J}{N} + 2\,\sigma_{max}^2 \frac{M}{N} \tag{18}$$

The first term on the right-side of Eq. (18) is the conventional Mean Squared Error (MSE). The second term is an Over-Fit Penalty (OFP) that prevents over-fitting of the model with too many terms, which is detrimental to model prediction accuracy [12]. The σ_{max}^2 in Eq. (18) is the maximum variance of elements in the error vector Ξ assuming the correct model structure. The factor of 2 in the Over-Fit Penalty (OFP) accounts for the fact that the PSE is being used when the model structure is not correct, that is, during the model structure determination stage.

The definition of PSE is very logical and elegant, but there are some differences in the literature in selecting proper value for σ_{max}^2. Refs. [8-9] assume

$$\sigma_{max}^2 = \frac{\sigma_0^2}{2} \quad \text{where, } \sigma_0^2$$

is the variance estimated from the output measurements generated from repeated wind tunnel runs at the same test condition:

$$\sigma_0^2 = \frac{1}{N}\sum_{i=1}^{N}(y_i - \bar{y})^2 \text{ and } \bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i \tag{19}$$

In Ref. [13], it was assumed that $\sigma_{max}^2 = 25\sigma_0^2$. Further, it was found in Ref. [13] that the model structure determined using PSE was virtually the same for σ_{max}^2 in the range $9\sigma_0^2 \le \sigma_{max}^2 \le 100\sigma_0^2$. This implies that for each wind tunnel database, a suitable value needs to be selected for σ_{max}^2 by trial and error. In general, PSE can be expressed as

$$PSE = \frac{J}{N} + K\,\sigma_0^2 \frac{M}{N} \tag{20}$$

where proper value for K has to be selected for a given wind tunnel database. In the current work, it was found that the value of K = 2 yielded adequate global models for F-16 aircraft.

The PSE criterion is evaluated as each orthogonal function is added to the proposed model with choice of the functions that cause the maximum reduction in the fit error. At some point, the PSE reaches a minimum and any further addition of orthogonal functions to the model causes the PSE to increase. Thus, the minimum in PSE defines an adequate model structure with good predictive capability.

Classical Gram-Schmidt Algorithm

The classical Gram-Schmidt procedure computes matrix W, one column at a time, from Eq. (6) and orthogonalizes P (at the k^t stage) by making the k^t column orthogonal to each of the (k-1) previously orthogonalized columns. The operation is repeated for k=2, ..., M. The computational procedure is represented as:

$$\begin{aligned} w_1 &= p_1 \\ a_{ik} &= \frac{\langle w_i, p_k \rangle}{\langle w_i, w_i \rangle} \text{ for } 1 \le i < k,\ k = 2, \dots, M \\ w_k &= p_k - \sum_{i=1}^{k-1} a_{ik} w_i \end{aligned} \tag{21}$$

where $\langle .,. \rangle$ denotes the inner product, that is,

$$\langle w_i, p_k \rangle = w_i^T p_k = \sum_{j=1}^{N} w_i(j)\, p_k(j)$$

Identifying Global Models for F-16 Aircraft

Wind tunnel aerodynamic data for a 16% scale model of the F-16 aircraft, flying at low Mach numbers (< 0.6), out of ground effect, with landing gear retracted and no external stores, is given in Ref. 16. A simplified version of the original wind tunnel database is given in Ref. 14. The simplified wind tunnel data is tabulated for angle-of attack range from -10 to 45 degrees, the sideslip angle range of ± 30 degrees, the elevator deflection range of ± 25 degrees, the ailerons deflection range of ± 21.5 degrees, and the rudder deflection range of ±30 degrees.

In the present work, the simplified wind tunnel database of F-16 was used to obtain global models of the aerodynamic force and moment coefficients. The effect of different weighting factors K in the OFP term was also investigated. Compared to the global models generated in Ref. 9 using a complex orthogonalisation scheme with $\sigma^2_{max}=\frac{\sigma_0^2}{2}$, the global models identified in the present investigations using a simpler approach with $\sigma^2_{max}=\sigma_0^2$ provided satisfactory match with the wind tunnel data.

Typical results of the aerodynamic global modeling for the F-16 wind tunnel database are provided in Tables 1-3 and Figs. 1-9. The PSE for $C_{xq}(\alpha)$ in Fig.1 indicates that the global model for the coefficient should have five terms.

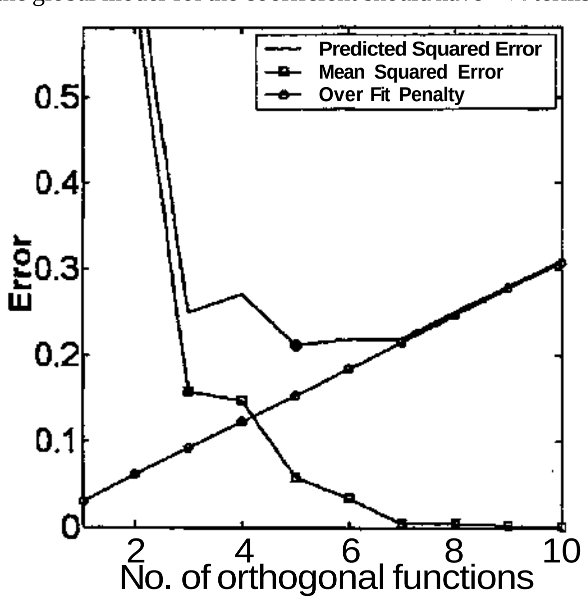

Fig. 1 Predicted squared error components for the coefficient $C_{xq}(\alpha)$

A similar observation is made in Ref. [9]. While the global model of $C_{xq}(\alpha)$ in Ref [9] has an MSE of 0.072668, the present model expressed in Table-1 and plotted in Fig.2 has an MSE of 0.05863873, and hence provides a better fit to the wind tunnel data. Fig. 3 shows that a global model with seven terms will fit the wind tunnel data better, but sucha a model may not have good predictive capability.

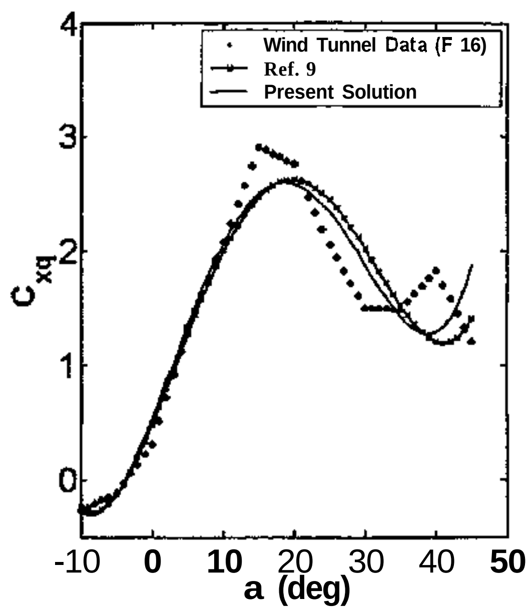

Fig. 2 Comparative plots for the coefficient $C_{xq}(\alpha)$ with Jive terms

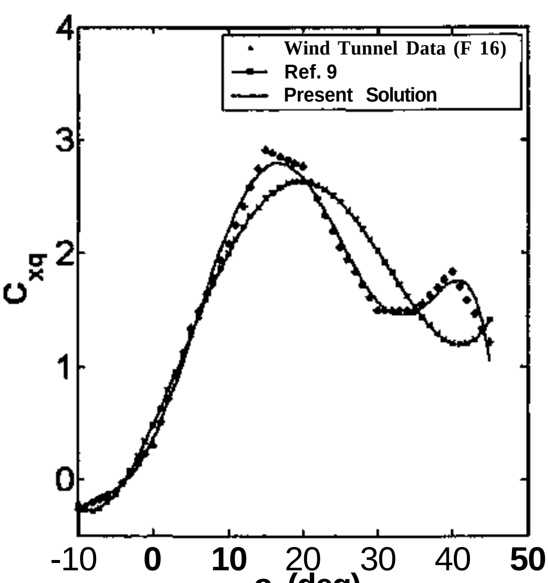

Fig. 3 Comparative plots for the coefficient $C_{xq}(\alpha)$ with seven terms

The PSE for $C_{zq}(\alpha)$ in Fig.4 indicates that the global model for the coefficient should include at least 5 terms. The model structure for $C_{zq}(\alpha)$ is given in Table-2. Fig.5 shows that the present solution and the model from Ref. [9] are comparable, and both fit the wind tunnel data equally well.

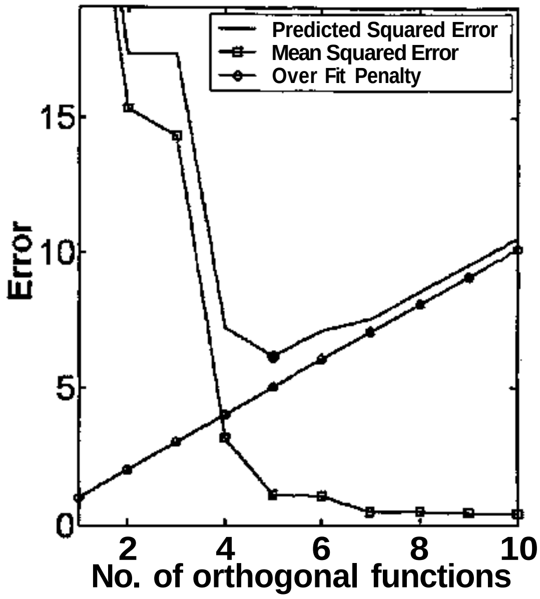

Fig. 4 Predicted squared error components for the coefficient $C_{zq}(\alpha)$

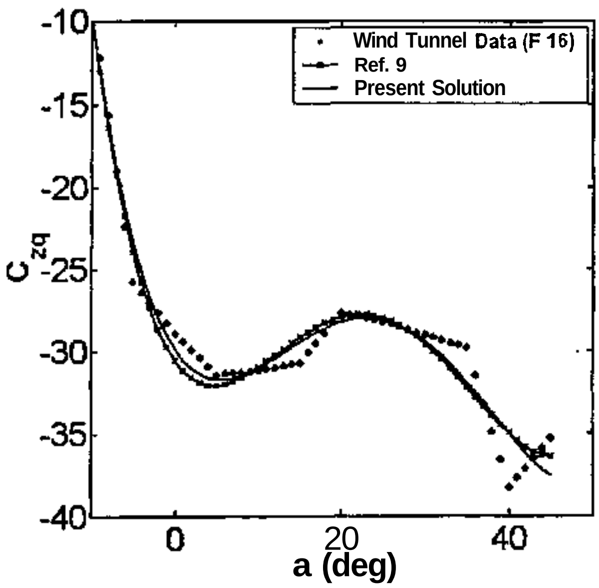

Fig. 5 Comparative plots for the coefficient $C_{zq}(\alpha)$

The error components for the function $C_l(\alpha, \beta)$ in Fig.6 show that it is not always possible to rely entirely on PSE metric to decide upon the number of terms to be included in the model. Going by the plots of PSE in Fig.6, including 4 terms in the global model should give a reasonable match with the wind tunnel data. However, past experience in modeling coefficients that are functions of two or more variables shows that more than four terms will be required to get a good fit for $C_l(\alpha, \beta)$ with the wind tunnel data. For the present case, it was found that at least 8 terms are required to model $C_l(\alpha$, (3) adequately. The global model for $C_l(\alpha$, P) in Ref. [9] and the present model structure defined in Table-3 have eight terms each. How-

Table: 1 Model Structure and Parameter Values for the Function $C_{xq}(\alpha)$

$C_{xq}(\alpha) = b_0 + b_1\alpha + b_2\alpha^2 + b_3\alpha^3 + b_4\alpha^4$

Sl. No.	Parameter	Ref. 9	Present Solution
1	b_0	0.4833383	0.5375464
2	b_1	8.644627	9.1225574
3	b_2	11.31098	9.7260248
4	b_3	-74.22961	-78.6050947
5	b_4	60.75776	68.9893810
Mean Squared Error		0.072668	0.05863873
Over-fit penalty		0.077109	0.15421976
Predicted squared error		0.149777	0.21285850

Table: 2 Model Structure and Parameter Values for the Function $C_{zq}(\alpha)$

$C_{zq}(\alpha) = g_0 + g_1\alpha + g_2\alpha^2 + g_3\alpha^3 + g_4\alpha^4$

Sl. No.	Parameter	Ref. 9	Present Solution
1	g_0	-30.54956	-29.8579836
2	g_1	-41.32305	-43.6810596
3	g_2	329.27880	306.1325795
4	g_3	-684.80380	-596.2637308
5	g_4	408.02440	332.7543198
Mean Squared Error		1.293684	1.12690974
Over-fit penalty		2.532696	5.06532924
Predicted squared error		3.826380	6.19223898

ever, the model for $C_1(\alpha, P)$ in Ref. [9] includes the terms β^2 and $\alpha^3\beta$ which are missing from the model defined in Table-3. On the other hand, the terms β^3 and $\alpha^3\beta^2$ are included in the current model structure to have a better fit with the wind tunnel data at higher sideslip angles. This interplay between the higher order terms of the independent variables can have considerable bearing on the results. Since $C_1(\alpha, P) = 0$ for $p = 0$ it is assumed that $C_1(\alpha, \beta) = \beta * f(\alpha, \beta)$.

As seen from Figs. 7 to 9, the higher order terms for P in the present case result in better matching of the present solution with wind runnel data. For the plots shown in Fig. 7, the MSE for the model of Ref. [9] is 0.00004717, while the MSE for the present global model is 0.00003596, indicating a 24% improvement over the model of Rs. [9]. Likewise, for p=25 deg. In Fig.8, the MSE value for the model of Ref. [9] is 0.00007234, and the MSE for the model from present analysis is 0.00002706, a 62% improvement over the model of Ref. [9]. For β=30 deg in Fig.9, the MSE for the model of Ref. [9] is 0.00013567 while that for the present solution is 0.00004525, indicating about 66% better match of the present solution with the wind tunnel data for the given range of independent variables.

Similar exercise of identifying global models was carried out for other aerodynamic force and moment coefficients, the results for which are not presented here for the sake of brevity. In all the cases, the identified global models were either comparable or better than the corresponding models given in Ref. [9].

Table: 3 Model Structure and Parameter Values for the Function $C_l(\alpha, p)$

$$C_l(\alpha, P) = \beta(h_{00} + h_{10}\alpha + h_{20}\alpha^2 + h_{01}\beta + h_{11}\alpha\beta + h_{30}\alpha^3 + h_{40}\alpha^4 + h_{21}\alpha^2\beta + h_{31}\alpha^3\beta + h_{02}\beta^2)$$

SI. No.	Parameter	Ref. 9	Present Solution
1	h_{00}	-0.10558583	-0.10064754
2	h_{10}	-0.5776677	-0.66422883
3	h_{20}	-0.01672435	1.76296703
4	h_{01}	0.1357256	
5	h_{11}	0.2172952	0.65765011
6	h_{30}	3.464156	---
7	h_{40}	-2.835451	-1.17821622
8	h_{21}	-1.098104	-4.19529581
9	h_{31}	---	3.36483413
10	h_{02}	—	0.24881440
Mean Squared Error		0.00005755	0.00003398
Over-fit penalty		0.00007178	0.00014356
Predicted squared error		0.00012933	0.00017754

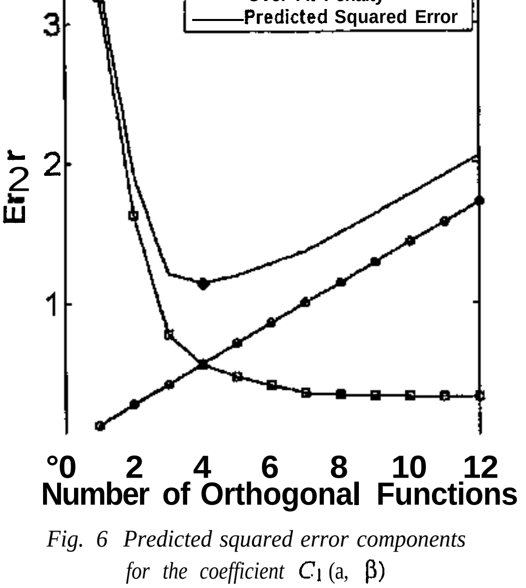

Fig. 6 Predicted squared error components for the coefficient C_l (a, β)

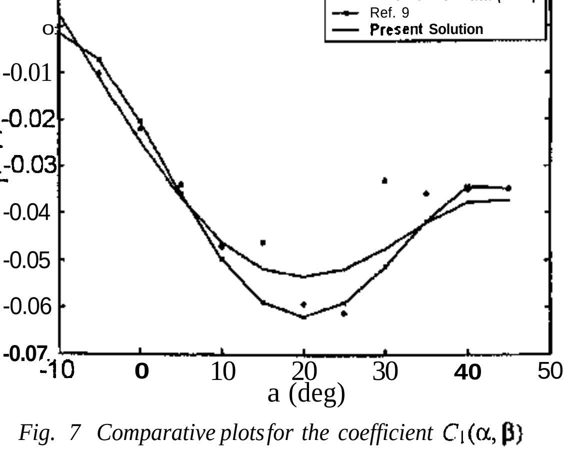

Fig. 7 Comparative plots for the coefficient $C_l(\alpha, \beta)$ for P = 20 deg

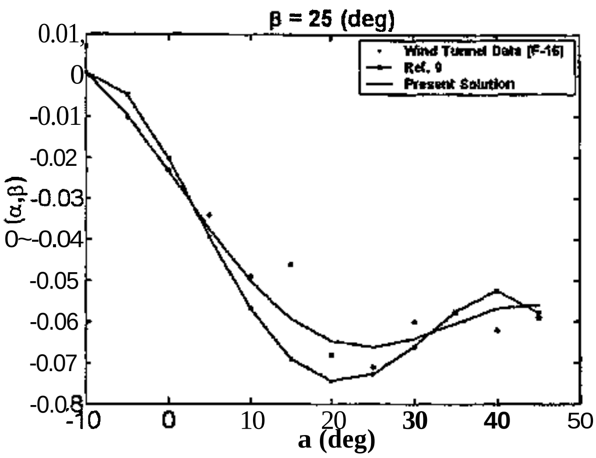

Fig. 8 Comparative plotsfor the coefficient $C_l(\alpha, \beta)$ *for* (3 *= 25 deg*

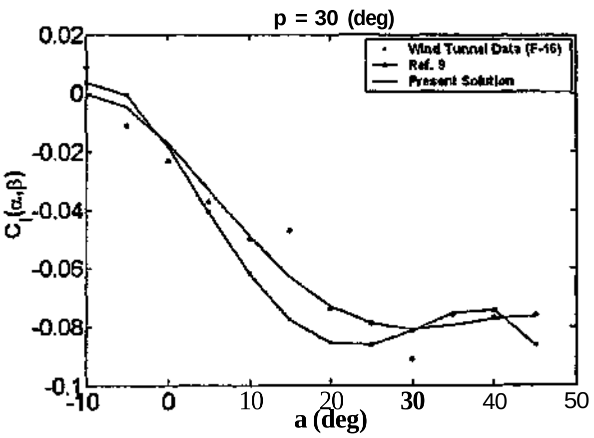

Fig. 9 Comparative plotsfor the coefficient $C_l(\alpha, \beta)$ *for* P *= 30 deg*

Conclusions

A simple technique based on classical Gram Schmidt method and Predicted Squared Error (PSE) metric is used to generate orthogonal functions to determine nonlinear aerodynamic global models of aircraft force and moment coefficients, from F-16 wind runnel data. A program code was written in MATLAB for this purpose. Results show that the agreement between the global models and the wind tunnel data is good. Comparison of the identified models with those given in Ref. [9], obtained by using a more complex scheme of orthogonal function generation, shows that the simpler approach used in the present analysis yields equally comparable or better global models.

Based on the current work, the following conclusions are made :

- The orthogonal function modeling technique offers a simple method for determining the model structure and estimating the parameters of a global model.
- The classical Gram-Schmidt orthogonalization procedure is adequate for generating the orthogonal modeling functions. Use of complex algorithms, based on modified Gram-Schmidt method, is not necessary.
- The PSE metric is a useful criterion for determining the model structure and the number of terms for a compact model. However, in certain cases, it might become necessary to include more terms than suggested by PSE to achieve a better fit to the wind tunnel data.
- To attain a good fit of the global model to a given wind tunnel database, it is necessary to select a proper value for the multiplier K, in the relationship :

$$PSE = \frac{J}{N} + K\,\sigma_0^2\,\frac{M}{N}$$

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